Complex dynamics in a nerve fiber model with periodic coefficients

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Abstract

We deal with the periodic boundary value problem for a second-order nonlinear ODE which includes the case of the Nagumo type equation $v_{xx} - gv + n(x)F(v) = 0$, previously considered by Grindrod and Sleeman and by Chen and Bell in the study of nerve fiber models. In some recent works we discussed the case of nonexistence of nontrivial solutions as well as the case in which many positive periodic solutions may arise, the different situations depending by threshold parameters related to the weight function $n(x)$. Here we show that for a step function $n(x)$ (or for small perturbations of it) it is possible to obtain infinitely many periodic solutions and chaotic dynamics, due to the presence of a topological horseshoe (according to Kennedy and Yorke).

Key words: Nagumo type equation, periodic solutions, subharmonics, chaotic like dynamics, topological horseshoe, Bernoulli shift.
2000 AMS subject classification : 34C25, 37E40, 92C20.

1 Introduction and statement of the main results

This paper deals with the study of a class of nonlinear second order ODEs which was proposed by Chen and Bell [3] as a model for a nerve fiber with spines. In their model, the authors consider a nerve fiber as an infinitely long cable. The spines, which are small protrusions on the membrane, are assumed to be periodically distributed. Their population is characterized by a density $n = n(x)$ and by a lumped ohmic resistance $1/\rho$ of the spine stem. The current flowing through the spine stem is given by $\rho(u - v)$, where $v(x,t)$ and $u(x,t)$ represent two membrane potentials (v is the one corresponding to the dendrite, while u is that at the spine head). Chen and Bell, following the cable theory approach previously considered by Baer and Rinzel [1] and simplifying the

active membrane dynamics from the Hodgkin-Huxley form to the FitzHugh-Nagumo's, derived the following system

$$\begin{cases} v_t = v_{xx} - gv + n(x)\rho(u - v) \\ u_t = \rho(u - v) + f(u), \end{cases}$$

where $g > 0$ is a constant such that $1/g$ represents the dendritic membrane resistance and f has the typical cubic form of the Nagumo equation [13]. The search of the steady state solutions $u = u(x), v = v(x)$ of the above system, lead to the study of

$$\begin{cases} v_{xx} - gv + n(x)f(u) = 0 \\ v = S(u; \rho) := u - \rho^{-1}f(u). \end{cases}$$

In [3], the authors discuss, separately, the cases in which the function S is invertible or not (for a given ρ). If S is invertible and therefore we can write $u = \tilde{R}(v)$, for a suitable monotone increasing function $\tilde{R}$, we can confine ourselves to the study of the nonlinear second order scalar ODE

$$v_{xx} - gv + n(x)F(v) = 0, \tag{1}$$

with $F(v) := f(\tilde{R}(v))$. We note that the possibility of inverting S and therefore considering equation (1) is not too restrictive and can be assumed when ρ is sufficiently large. In [3] the function $n(x)$ is supposed to be piecewise constant and β-periodic. More precisely, two positive constants n_0 (small) and n_1 (large) are taken such that

$$n(x) = \begin{cases} n_1, & \text{for } 0 \leq x < \alpha, \\ n_0, & \text{for } \alpha \leq x < \beta. \end{cases}$$

In [3, Lemma 4.1], assuming further a condition which implies that $\int_0^1 F(s)\,ds > 0$, it is stated that equation (1) has a positive periodic solution when α is sufficiently large (i.e., α is close to β).

In some recent papers [20,21] we extended the above recalled Chen and Bell result toward different directions. In particular, in [20], we proved (via a variational approach) that the existence of at least two positive periodic solutions is guaranteed for an arbitrary weight function $n(x)$, provided that $\int_0^\beta n(x)\,dx$ is sufficiently large. On the other hand, in [21], we obtained (via the Poincaré-Birkhoff fixed point theorem) the existence of many positive periodic solutions under more restrictive hypotheses on $n(x)$ (but general enough to include the piecewise constant case) and removing the requirement that $\int_0^1 F(s)\,ds > 0$. In this work we come back to the original assumptions on $n(x)$ of Chen and

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Bell's paper with the aim to prove that their model presents solutions with a very complicated behavior.

In order to introduce our results, we consider equation (1), where, throughout the paper, $g > 0$, $n : \mathbb{R} \to \mathbb{R}^+ := [0, +\infty)$ is a β-periodic locally integrable function and $F : \mathbb{R} \to \mathbb{R}$ is a C^1 function with a N-shaped graph. More precisely, we assume that there exists $a \in]0, 1[$ such that

$$F(0) = F(a) = F(1) = 0$$

and, moreover,

$$F(s) < 0 \ \text{ for } \ 0 < s < a \ \text{ and } \ s > 1, \quad F(s) > 0 \ \text{ for } \ s < 0 \ \text{ and } \ a < s < 1.$$

A typical function F satisfying such conditions is given by the cubic nonlinearity

$$F(s) = s(s - a)(1 - s)$$

of the celebrated FitzHugh-Nagumo equation [8,9]. We look for solutions $v(x)$ of (1) with

$$0 < v(x) < 1, \quad \forall x \in \mathbb{R}$$

and, in particular, we shall focus our attention to the search of periodic solutions as well as to the detection of solutions presenting some kind of chaotic-like behavior.

Starting with the 70's a great deal of theoretical and numerical results have been concerned the investigation of nonlinear ordinary and partial differential equations and systems associated to the FitzHugh-Nagumo's. Variational or topological functional analytic methods have been successfully applied in the study of various boundary value problems associated to such equations. With this respect, we also recall some recent works by Sweers-Troy [18] and Dancer-Yan [4,5] where an abundance of solutions with a peculiar behavior for the Dirichlet or Neumann BVPs is established for related (although different from ours) Nagumo type systems. We stress, however, that some special features of equation (1), like the presence of a periodic weight, makes natural to address our investigation toward different aspects that, perhaps, have not been yet fully discussed in the literature.

In the present paper we propose a dynamical system approach. To this end, we think at the one-dimensional x-variable (representing the longitudinal axial dimension of the idealized nerve fiber) as a temporal variable, in order to deal with a first order system in the plane of the form

$$\begin{cases} v' = y \\ y' = gv - n(t)F(v). \end{cases} \tag{2}$$

We denote by ψ the Poincaré's operator related to system (2), that is the map which associates to a point $z \in \mathbb{R}^2$ the value $\psi(z) := \zeta(\beta, z)$ of the solution $\zeta(\cdot) = \zeta(\cdot, z)$ of system (2) with $\zeta(0) = z$. We notice that, without further assumptions on F, we don't have ψ necessarily defined on the whole plane $\mathbb{R}^2$. On the other hand, by the smoothness of F, we know that ψ is a (orientation-preserving) homeomorphism of its domain $D_\psi \subseteq \mathbb{R}^2$ onto its image $R_\psi \subseteq \mathbb{R}^2$.

Our main result is the following:

Theorem 1.1 *Assume that $F(\cdot)$ is strictly convex on $[0, a]$, strictly concave on $[a, 1]$ and satisfies*

$$\int_0^1 F(s)\,ds > 0. \tag{3}$$

Define

$$m_0^* := g \Big/ \Big(\max_{s \in [a,1]} \frac{F(s)}{s} \Big). \tag{4}$$

Let $0 < \alpha < \beta$ and suppose that $n(\cdot)$ is the stepwise function given by

$$n(t) := \begin{cases} n_1, & \text{for } 0 \le t < \alpha, \\ n_0, & \text{for } \alpha \le t < \beta, \end{cases} \tag{5}$$

with $0 < n_0 < n_1$. Then there exists $\omega^* > m_0^*$ such that for every n_0 and n_1 with

$$0 < n_0 < m_0^* < \omega^* < n_1,$$

there exist a compact invariant set

$$\Lambda \subseteq \,]a, 1[\times \,]0, +\infty)$$

for which $\psi|_\Lambda$ is semiconjugate via a continuous surjection h to a two-sided Bernoulli shift on two symbols. Moreover, for each periodic sequence of two symbols $(i_k)_{k \in \mathbb{Z}}$ there is a point $z \in \Lambda$ which is periodic and such that $h(z) = (i_k)_{k \in \mathbb{Z}}$.

As we shall see at the end of the proof of Theorem 1.1 (which is performed in Section 4) the semiconjugation of $\psi|_\Lambda$ allows the following interpretation in terms of the solutions to (1):

Consider a two-sided sequence of two symbols $\xi := (i_k)_{k \in \mathbb{Z}}$ with

$$i_k \in \{1, 2\}, \quad \forall\, k \in \mathbb{Z}.$$

Then, there is at least one solution $v_\xi = v_\xi(x)$ of (1) which is defined for every $x \in \mathbb{R}$ and satisfies

$$0 < v_\xi(x) < 1, \quad \forall\, x \in \mathbb{R}$$

as well as $(v_\xi(0), \frac{d}{dx}v_\xi(0)) \in \Lambda$. Moreover, the symbol $i_k = 1$ means that $v_\xi(x)$ has precisely two strict maximum points separated by one strict minimum point

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along the time interval $[(k-1)\beta, (k-1)\beta + \alpha]$, while, the symbol $i_k = 2$ means that $v_\xi(x)$ has precisely three strict maximum points separated by two strict minimum points along the time interval $[(k-1)\beta, (k-1)\beta + \alpha]$. In both the situations, $v_\xi(x)$ is convex in the interval $[(k-1)\beta + \alpha, k\beta]$, with $v'_\xi((k-1)\beta + \alpha) < 0$ and $v'_\xi(k\beta) > 0$. If the sequence $(i_k)_{k\in\mathbb{Z}}$ is periodic, that is $i_k = i_{k+\ell}$ for some $\ell \geq 1$, then we can take $v_\xi(\cdot)$ as a $\ell\beta$-periodic solution as well.

A minor modification in the argument of the proof of Theorem 1.1 yields to the following result.

Theorem 1.2 *Assume that $F(\cdot)$ is strictly convex on $[0, a]$, strictly concave on $[a, 1]$ and satisfies (3). Let m_0^* be like in (4) and set*

$$m_1^* := \frac{g}{2\int_0^1 F(s)\,ds}.$$

Suppose that $n(\cdot)$ is defined as in (5) with

$$0 < n_0 < m_0^* < m_1^* < n_1.$$

Then there exist $\alpha^* > 0$ and $\delta > 0$ such that for every α and β with

$$\alpha^* < \alpha < \beta < \alpha + \delta,$$

the same conclusion of Theorem 1.1 holds.

We observe that it is possible to obtain extensions of Theorem 1.1 and Theorem 1.2 by producing symbolic dynamics on p objects with $p > 2$ (see Remark 4.1).

2 Notation and basic tools

Throughout the paper we denote by $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{R}_0^+$ the sets of real, real nonnegative and positive numbers. By $||\cdot||$ me mean a given norm (for instance, the euclidean one) in $\mathbb{R}^2$.

In the sequel we consider some special subsets of the plane, called *generalized rectangles*. By such a name we mean any subset of $\mathbb{R}^2$ which is homeomorphic to the unit square $[0, 1]^2$. It is possible to define an orientation for a generalized rectangle $\mathcal{R}$, by selecting two disjoint compact sub-arcs $\mathcal{R}_l^-$ and $\mathcal{R}_r^-$ of its boundary, called the left and the right sides of $\mathcal{R}$. In a more formal way, using the Jordan-Shoenflies theorem, we can choose a homeomorphism h of

the plane onto itself such that

$$h([0,1]^2) = \mathcal{R}, \quad h(\{0\} \times [0,1]) = \mathcal{R}_l^- \text{ and } h(\{1\} \times [0,1]) = \mathcal{R}_r^- .$$

In this situation, we also set

$$\mathcal{R}^- := \mathcal{R}_l^- \cup \mathcal{R}_r^- .$$

By a path γ we mean a continuous map $\gamma : [0,1] \to \mathbb{R}^2$.

Let $(\mathcal{A}, \mathcal{A}^-)$ and $(\mathcal{B}, \mathcal{B}^-)$ be oriented rectangles and suppose that $\psi : \mathbb{R}^2 \supseteq D_\psi \to \mathbb{R}^2$ is a map which is continuous on a set $\mathcal{D} \subseteq D_\psi \cap \mathcal{A}$. We say that $(\mathcal{D}, \psi)$ *stretches* $(\mathcal{A}, \mathcal{A}^-)$ to $(\mathcal{B}, \mathcal{B}^-)$ *along the paths* and write

$$(\mathcal{D}, \psi) : (\mathcal{A}, \mathcal{A}^-) \diamond\!\!\!\!\rightsquigarrow (\mathcal{B}, \mathcal{B}^-),$$

if, for every path $\gamma : [0,1] \to \mathcal{A}$ with $\gamma(0) \in \mathcal{A}_l^-$ and $\gamma(1) \in \mathcal{A}_r^-$, there exists a subinterval $[t_1, t_2] \subseteq [0,1]$ such that

$$\gamma(t) \in \mathcal{D}, \quad \psi(\gamma(t)) \in \mathcal{B}, \quad \forall\, t \in [t_1, t_2]$$

and $\psi(\gamma(t_1))$, $\psi(\gamma(t_2))$ belong to different components of $\mathcal{B}^-$.

It is easy to check that this property of stretching along the paths is preserved under the compositions of maps. Moreover, as proved in [15,16], if

$$(\mathcal{D}, \psi) : (\mathcal{R}, \mathcal{R}^-) \diamond\!\!\!\!\rightsquigarrow (\mathcal{R}, \mathcal{R}^-),$$

then there exists at least one fixed point for ψ in $\mathcal{D}$. Such a fixed point property, when applied to different subsets of the domain and to the iterates of ψ, may be exploited in order to find many different periodic points for the map ψ.

The key tool that we use for the proof of the existence of chaotic-like dynamics for the Poincaré's map associated to equation (1) is the following lemma which combines the above mentioned fixed point theorem with the theory of topological horseshoes developed by Kennedy and Yorke in [11] (see also [10]).

Lemma 2.1 *Let $(\mathcal{R}, \mathcal{R}^-)$ be an oriented rectangle and let $\psi : \mathbb{R}^2 \supseteq D_\psi \to \mathbb{R}^2$ be a map. Assume there exist compact sets $\mathcal{D}_i \subseteq \mathcal{R} \cap D_\psi$ (for $i = 1, 2$) with*

$$\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset,$$

such that
$$(\mathcal{D}_i, \psi) : (\mathcal{R}, \mathcal{R}^-) \diamond\!\!\!\!\rightsquigarrow (\mathcal{R}, \mathcal{R}^-), \quad \forall\, i = 1, 2.$$
Then the following conclusions hold:

- *For any two−sided sequence of two symbols $(i_k)_{k \in \mathbb{Z}} \in \Sigma_2 := \{1, 2\}^\mathbb{Z}$, there exists a sequence $(w_k)_{k \in \mathbb{Z}}$ such that $w_k \in \mathcal{D}_{i_k}$ and $\psi(w_k) = w_{k+1}$, for all $k \in \mathbb{Z}$;*

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- *If the sequence $(i_k)_{k \in \mathbb{Z}} \in \Sigma_2$ is m-periodic ($m \geq 1$), then there exists a corresponding sequence $(w_k)_{k \in \mathbb{Z}}$ with $w_k \in \mathcal{D}_{i_k}$ which is m-periodic.*

Furthermore, as a consequence of the above properties, there exists a nonempty compact set

$$\Lambda \subseteq \mathcal{D}_1 \cup \mathcal{D}_2$$

which is invariant for ψ and such that $\psi|_\Lambda$ is semiconjugate to the two-sided Bernoulli shift on two symbols. The subset of Λ made by the periodic points of ψ is dense in Λ and the counterimage (by the semiconjugacy) of any periodic sequence in Σ_2 contains a periodic point of ψ.

For the proof, see [15,16] and [17] for more recent details. Note that the semi-conjugation to the Bernoulli shift and the density of periodic points are typical requirements for chaotic dynamics. The kind of chaotic-like dynamics described by Lemma 2.1 is based on the definition of chaos in the coin-tossing sense by Kirchgraber and Stoffer [12]. The same chaotic behavior is also obtained by Kennedy, Koçak e Yorke in [10, Proposition 5]. With respect to [10] and [12], our case takes into account also of the presence of periodic itineraries generated by periodic points. Concerning the applications to second order ODEs with periodic coefficients, where ψ is the Poincaré map associated to an equivalent first order differential system in the plane, the complex dynamics we obtain is in line with similar results appeared in the literature for different type of equations (see, for instance Capietto, Dambrosio and Papini [2]). Finally, we refer to Mischaikow and Mrozek [14] and to Zgliczyński and Gidea [22] for related topological approaches in higher dimension.

3 Technical lemmas

Let us consider the equation

$$x'' - gx + n(t)F(x) = 0 \qquad (x' := \frac{dx}{dt}), \tag{6}$$

as well as the associated first order system in the phase-plane

$$\begin{cases} x' = y \\ y' = gx - n(t)F(x), \end{cases} \tag{7}$$

where $n(t) \geq 0$ is a β-periodic piecewise continuous function that will be defined in a more precise manner in the sequel.

The map $F : \mathbb{R} \to \mathbb{R}$ is a C^1 function with a N-shaped graph. In particular,

we assume there exists $a \in \,]0, 1[$ such that

$$F(0) = F(a) = F(1) = 0$$

and, moreover,

$F(s) < 0$ for $0 < s < a$ and $s > 1,$ $F(s) > 0$ for $s < 0$ and $a < s < 1.$

A typical example for F is given by the Nagumo type cubic nonlinearity

$$F(s) = s(s - a)(1 - s). \tag{8}$$

For the moment and in order to simplify the subsequent discussion, we assume (8). We point out, however, that the properties we are going to present below are still true for a broader class of functions.

We are looking for solutions $x(\cdot)$ to (6) such that

$$0 \leq x(t) \leq 1, \quad \forall\, t \in \mathbb{R}. \tag{9}$$

Accordingly, these solutions are the same also for any other equation where the function $F(s)$ is the same like the given one in $[0, 1]$ but possibly different elsewhere. For convenience we then suppose that

$$|F(s)| \leq 1, \quad \forall\, s \in \mathbb{R} \setminus [0, 1]$$

so that all the solutions of equation (6) are globally defined. Clearly, at the conclusion of our argument, we need to check that the solutions we are interested in satisfy the constraint in (9). To this purpose, we state the following lemma whose proof (which is valid also for a more general class of equations) is postponed in the Appendix.

Lemma 3.1 *Let $n : [t_0, t_1] \to \mathbb{R}^+$ be a Lebesgue integrable function and suppose that $F_0 : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz function such that*

$F_0(s) > 0$ *for* $s < 0,$ $F_0(s) < 0$ *for* $s > 1$ *and* $F_0(s) = F(s),$ $\forall\, s \in [0, 1].$

Let $x(\cdot)$ be a solution of

$$x'' - gx + n(t)F_0(x) = 0, \quad (g \geq 0)$$

defined on $[t_0, t_1]$ and such that

$$0 < x(t_0), x(t_1) < 1.$$

Then,
$$0 < x(t) < 1, \quad \forall\, t \in [t_0, t_1]$$
and, therefore, $x(\cdot)$ is a solution of (6).

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As a preliminary discussion, we start by a phase-plane analysis of the trajectories of (7) in the case when $n(\cdot) = \text{constant} = \mu > 0$. Accordingly, we study the autonomous system

(E)
$$\begin{cases} x' = y \\ y' = gx - \mu\, F(x), \end{cases}$$

and look at the effects of the parameter μ on the qualitative behavior of the orbits.

First of all we observe that there is m_0^* such that, for $\mu \in\,]0, m_0^*[$

$$gs - \mu F(s) > 0, \quad \forall\, s > 0,$$

while for each $\mu > m_0^*$ there are two nontrivial equilibrium points $P = (a_\mu, 0)$ and $Q = (c_\mu, 0)$ to system (E) with

$$0 < a < a_\mu < c_\mu < 1$$

(see, for instance, Fig. 1 for an example which depicts this situation). Moreover, $a_\mu \to a^+$ and $c_\mu \to 1^-$ as $\mu \to +\infty$. The point P is a center and Q is a saddle. This is true for $F(s)$ defined in (8) as well as for any function F which is strictly concave in $[a, 1]$.

The actual value of m_0^* can be computed as

$$m_0^* = \frac{g}{F'(s^*)}\,,$$

where $s^* \in\,]a, 1[$ solves the equation $sF'(s) = F(s)$ yielding the maximal slope.

System (E) is of conservative type with energy

$$\mathcal{E}(x, y) = \mathcal{E}^\mu(x, y) := \frac{1}{2}y^2 - g\frac{1}{2}x^2 + \mu\mathcal{F}(x), \tag{10}$$

where
$$\mathcal{F}(x) := \int_0^x F(s)\, ds.$$

We denote by L_c the part of the energy line at level c lying in the strip $0 \leq x \leq 1$, that is

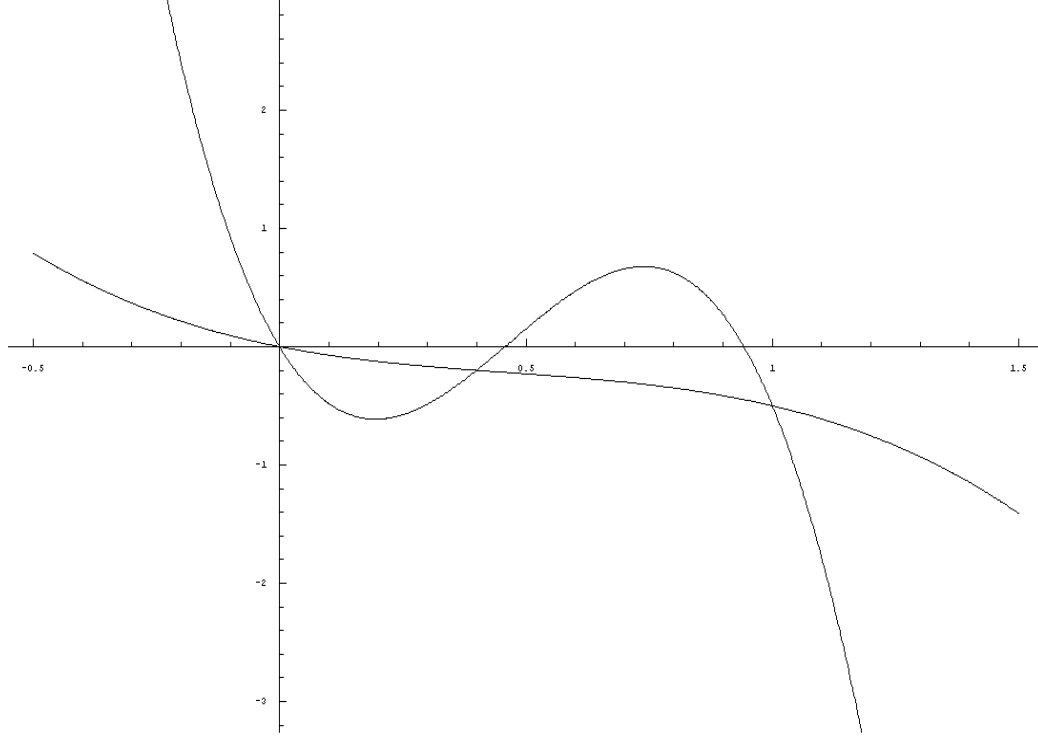$$L_c := \{(x, y) : \mathcal{E}(x, y) = c,\ 0 \leq x \leq 1\}.$$

Fig. 1. *Different shapes for the graph of the function* $y = -gs + \mu F(s)$, *for a small and a large value of* μ. *In this example, we have chosen* $g = 0.5$ *and* $F(x) = x(1-x)(x-a)$, *with* $a = 0.4$. *The graph of the decreasing function corresponds to the case* $\mu = n_0 = 0.8$, *while the other graph (with two humps) is obtained for* $\mu = n_1 = 16$.

We present now some technical results which provide dynamical information about the trajectories of system (E) for different values of the parameter μ.

Lemma 3.2 *Suppose that* $\mu \in]0, m_0^*[$. *Then, for every* $w_0 = (p_0, 0)$ *with* $0 \leq p_0 < 1$ *the level line* L_c *(c := $\mathcal{E}(w_0)$) passing through* w_0 *intersects any vertical line* $x = \xi$ *with* $p_0 < \xi \leq 1$ *exactly at two points* $(\xi, -y_\xi)$, (ξ, y_ξ). *For* $0 < p_0 < 1$, L_c *is an orbit path and the time needed to move from* $(\xi, -y_\xi)$ *to* w_0 *(which coincides to the time needed to move from* w_0 *to* (ξ, y_ξ)*) along the orbit is given by*

$$\sigma(p_0, \xi) = \int_{p_0}^{\xi} \frac{ds}{\sqrt{g(s^2 - p_0^2) - 2\mu(\mathcal{F}(s) - \mathcal{F}(p_0))}}.$$

Assume that

(H_0)
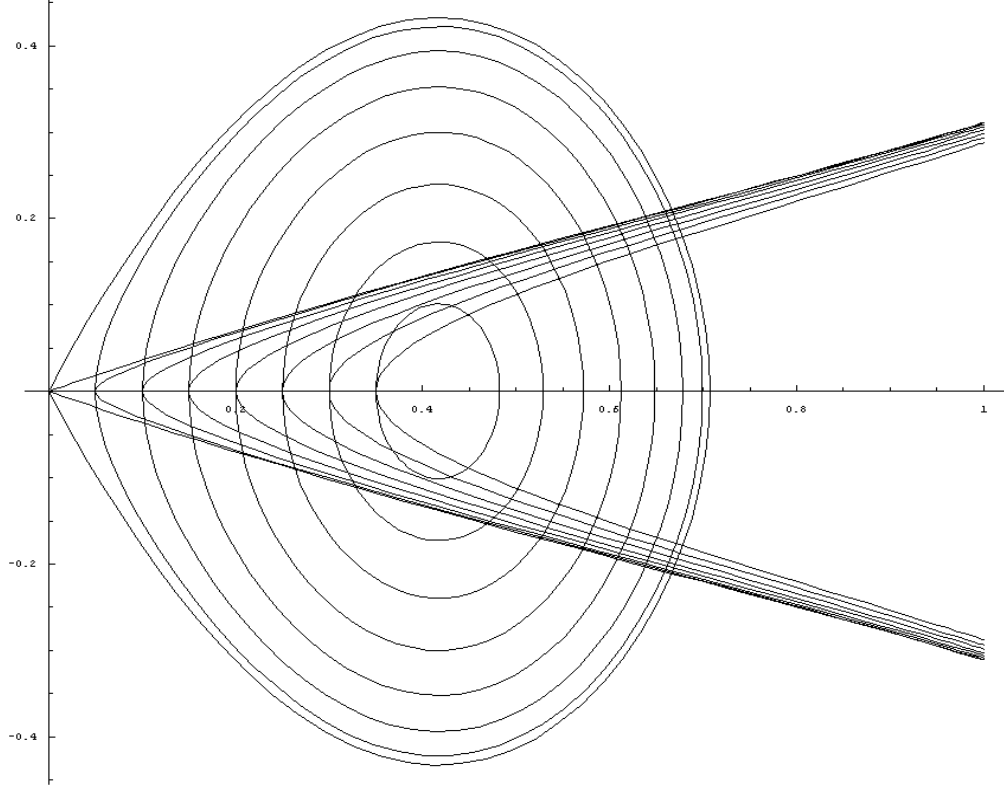$$\int_0^1 F(s)\, ds > 0.$$

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Fig. 2. *Level lines (at different energy) of system (E) with $\mu = 10$ (the closed lines) overlapped with those of the same system for $\mu = 0.1$. For this example, we have chosen $g = 0.1$ and $F(x) = x(1-x)(x-a)$, with $a = 0.4$.*

Then, there is $m_1^* > m_0^*$ such that, for each $\mu > m_1^*$, the level line L_0 is a homoclinic orbit of system (E) intersecting the abscissa at a point $(b_\mu, 0)$ with

$$a_\mu < b_\mu < c_\mu \, .$$

Moreover, for each $c \in]\mathcal{E}(P), 0[$, the level line L_c is a closed curve which corresponds to a periodic orbit of system (7) having minimal period

$$\tau(x_0) = \frac{2}{\sqrt{\mu}} \int_{x_0}^{x_1} \frac{ds}{\sqrt{2(\mathcal{F}(x_0) - \mathcal{F}(s)) - \frac{g}{\mu}(x_0^2 - s^2)}} \, ,$$

where x_0 and x_1, satisfying

$$0 < x_0 < a_\mu < x_1 < b_\mu \, ,$$

are the two solutions of the equation $\mathcal{E}(x, 0) = c$ with $x \in]0, 1[$.

Proof. We start by considering system (E) for $\mu \in]0, m_0^*[$. Let L_c be a level line passing through a point $w_0 = (p_0, 0)$ with $0 \le p_0 < 1$. By definition, we

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have
$$y^2 = \mathcal{G}(x) := g(x^2 - p_0{}^2) - 2\mu\big(\mathcal{F}(x) - \mathcal{F}(p_0)\big), \quad x \in [0, 1].$$
The function $\mathcal{G}$ is strictly increasing on $[0, 1]$. In fact $\mathcal{G}'(x) = 2(gx - \mu F(x)) > 0, \, \forall\, x \in\,]0, 1]$. Since $\mathcal{G}(p_0) = 0$, it turns out that $\sqrt{\mathcal{G}(x)}$ is defined on $[p_0, 1]$ and therefore L_c intersects any vertical line $x = \xi$ (for $p_0 < \xi \le 1$) exactly at the two points $(\xi, -y_\xi)$ and (ξ, y_ξ) with $y_\xi = \sqrt{\mathcal{G}(\xi)}$. From the relation

$$\frac{x'(t)}{\sqrt{\mathcal{G}(x(t))}} = 1,$$

which is satisfied for any time interval I in which $y(t) = x'(t) > 0$, we can compute the time $\sigma(p_0, \xi)$ at which a solution of system (E) (starting at the point w_0 for $t = 0$) reaches the point (ξ, y_ξ). Actually, this is the standard time-mapping formula which reads in our case as

$$\sigma(p_0, \xi) = \int_0^{\sigma(p_0,\xi)} \frac{x'(t)}{\sqrt{\mathcal{G}(x(t))}}\, dt = \int_{p_0}^{\xi} \frac{dx}{\sqrt{\mathcal{G}(x)}}.$$

This proves the first part of our lemma.

We consider now system (E) for $\mu > m_0^*$. In this case, we have two equilibrium points $P = (a_\mu, 0)$ and $Q = (c_\mu, 0)$, with $a < a_\mu < c_\mu < 1$. The level line L_0 passing through the origin is defined by the relation

$$y^2 = \mathcal{H}(x) := gx^2 - 2\mu\mathcal{F}(x), \quad x \in [0, 1].$$

The function $\mathcal{H}$ is strictly increasing and positive on $]0, a_\mu]$, strictly decreasing on $[a_\mu, c_\mu]$ and strictly increasing on $[c_\mu, 1]$. Hence, $\mathcal{H}(1) = g - 2\mu\mathcal{F}(1) > gc_\mu^2 - 2\mu\mathcal{F}(c_\mu) = \mathcal{H}(c_\mu)$. Since $\mathcal{F}(1) = \int_0^1 F(s)\, ds > 0$ (by assumption (H_0)) we know that for μ sufficiently large, say $\mu > \frac{g}{2\mathcal{F}(1)}$, the function $\mathcal{H}$ has exactly one zero in $]0, c_\mu[$ and such zero, denoted by b_μ, is contained in $]a_\mu, c_\mu[$. Thus, if we define m_1^* as

$$m_1^* := \frac{g}{2 \int_0^1 F(s)\, ds},$$

we have that for every $\mu > m_1^*$, the level line L_0 is a close curve (the locus of a homoclinic trajectory of (E)) intersecting the abscissa at the point $(b_\mu, 0)$, with $a_\mu < b_\mu < c_\mu$. A similar argument shows that, for every $c \in\,]\mathcal{E}(P), 0[$, the level line L_c is a closed curve surrounding the equilibrium point P and, as a matter of fact, it is the locus of a periodic orbit.

In this case, the energy relation reads as

$$y^2 = 2c + \mathcal{H}(x), \quad x \in [0, 1].$$

The equation $\mathcal{H}(x) = -2c$, with $c \in\,]\mathcal{E}(P), 0[$ and $x \in\,]0, 1[$ has exactly two solutions, x_0 and x_1, with

$$0 < x_0 < a_\mu < x_1 < b_\mu.$$

We rewrite now the energy relation as

$$y^2 = -\mathcal{H}(x_0) + \mathcal{H}(x) = \mu\Big(2(\mathcal{F}(x_0) - \mathcal{F}(x)) - \frac{g}{\mu}(x_0^2 - x^2)\Big).$$

Arguing like in the first part of this proof (the case of μ small) we can compute again the time-mapping related to the orbit L_c. We find that the fundamental period $\tau(x_0)$ of such orbit is twice the time to move along L_c from $(x_0, 0)$ to $(x_1, 0)$ in the upper half plane $y = x' > 0$. Thus, integrating

$$\frac{x'(t)}{\sqrt{-\mathcal{H}(x_0) + \mathcal{H}(x(t))}} = 1,$$

we easily obtain the expression for $\tau(x_0)$. This concludes the proof.   □

Remark 3.1 *The choice of m_1^* given in the proof of Lemma 3.2 is not the optimal one. In fact, one could define m_1^* as the infimum of the $\mu > m_0^*$ such that the level line L_0 intersects the abscissa at some point in $]a_\mu, c_\mu[$.*

The next two lemmas are stated without the corresponding proofs which can be inferred from Lemma 3.2 via elementary considerations.

Lemma 3.3 *Let us define*

$$\Lambda := \sup_{0 < s \le 1} \frac{F(s)}{s}, \quad \Theta := \sup_{0 < s \le 1} \frac{-F(s)}{s}.$$

Suppose that $\mu \in]0, m_0^*[$. Then, for every p_0 with $0 < p_0 < 1$ and ξ with $p_0 < \xi \le 1$ we have

$$\frac{1}{\sqrt{g + \mu\Theta}} \cosh^{-1}(\tfrac{\xi}{p_0}) \le \sigma(p_0, \xi) \le \frac{1}{\sqrt{g - \mu\Lambda}} \cosh^{-1}(\tfrac{\xi}{p_0}).$$

Note that for F like in (8) it holds that $\Theta = |F'(0)|$. Moreover, $g - \mu\Lambda > 0$ for $\mu < m_0^*$.

Lemma 3.4 *Assume (H_0) and let $\mu > m_1^*$ (with m_1^* as in Lemma 3.2). Then, for every x_0 with*

$$0 < x_0 < a,$$

it follows that

$$\lim_{\mu \to +\infty} \tau(x_0)\sqrt{\mu} = \sqrt{2} \int_{x_0}^{x_0^+} \frac{ds}{\sqrt{\mathcal{F}(x_0) - \mathcal{F}(s)}},$$

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where $x_0^+ \in]a, 1[$ is such that

$$\int_{x_0}^{x_0^+} F(s)\, ds = 0.$$

At this point, we come back to the non-autonomous system (7) and define the function $n(t)$ as in [3]. Accordingly, we set

$$n(t) := \begin{cases} n_1, & \text{for } 0 \le t < \alpha, \\ n_0, & \text{for } \alpha \le t < \beta, \end{cases} \tag{11}$$

with

$$0 < n_0 < n_1.$$

In the sequel we treat n_0 as a small number and n_1 as a large one. More precisely, we shall always assume $0 < n_0 < m_0^* < m_1^* < n_1$. However, some further conditions on n_0 and n_1 will be required when needed.

In order to study equations (6) and (7) we analyze the system

$$(E_i) \qquad \qquad \begin{cases} x' = y \\ y' = gx - n_i\, F(x), \end{cases} \qquad i = 0, 1$$

and follow its trajectories for $i = 1$ along a time interval of length α and for $i = 0$ along a time interval of length $\beta - \alpha$. Clearly, (E_0) and (E_1) are nothing but two aspects of the previously studied system (E) with $\mu = n_0$ or $\mu = n_1$, respectively. Also the energy function $\mathcal{E} = \mathcal{E}^\mu$ defined in (10) takes two different (but similar) forms that we denote by $\mathcal{E}_0$ and $\mathcal{E}_1$, i.e.,

$$\mathcal{E}_i(x, y) = \mathcal{E}^{n_i}(x, y) = \frac{1}{2}y^2 - g\frac{1}{2}x^2 + n_i\mathcal{F}(x), \quad i = 0, 1.$$

The corresponding level lines L_c^i (contained in the strip $[0, 1] \times \mathbb{R}$) are defined consequently.

Let us fix a value

$$c \in]\mathcal{E}_1(a_{n_1}, 0), 0[$$

and consider the compact annular region

$$\mathcal{M}_c^1 := \{(x, y) : 0 \le x \le 1,\ c \le \mathcal{E}_1(x, y) \le 0\}$$

surrounding the point

$$P_1 := (a_{n_1}, 0).$$

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The set $\mathcal{M}_c^1$ is invariant for the dynamical system generated by (E_1) and, therefore, we can use the Prüfer transformation and express the solutions of (E_1) with initial value in $\mathcal{M}_c^1$ using polar coordinates with center at P_1. Accordingly, we define by $\theta(t,z)$ and $\rho(t,z)$ the angular and the radial coordinates of the solution $(x(t), y(t))$ of (E_1) with $(x(0), y(0)) = z \in \mathcal{M}_c^1$.

For each point z in the set

$$\mathcal{N}_c := \{(x,y) : a_{n_1} \leq x \leq 1, \ y \geq 0, \ c \leq \mathcal{E}_1(x,y) \leq 0\} = \mathcal{M}_c^1 \cap ([a_{n_1}, +\infty) \times [0, +\infty))$$

we can fix the angular coordinate in order to have

$$\theta(0, z) \in [0, \tfrac{\pi}{2}].$$

The next result rephrases the conclusions from Lemma 3.2 and Lemma 3.4 in terms of the angular coordinate. Indeed, we have:

Lemma 3.5 *Assume (H_0) and let $n_1 > m_1^*$. Then the following properties hold:*

- $\theta(\alpha, z) > -\pi, \quad \forall z \in L_0 \cap \mathcal{N}_c$;
- *for each $\varepsilon > 0$, for each $k \in \mathbb{N}$ and for each $c \in [-\varepsilon, 0[$, there is $n^* = n_{k,c}^*$, such that for every $n_1 \geq n^*$, $\theta(\alpha, z) < -\frac{\pi}{2} - 2k\pi, \quad \forall z \in L_c \cap \mathcal{N}_c$.*

Proof. Assumption (H_0) and the choice $n_1 > m_1^*$ ensure that the level line L_0 is homoclinic to the origin for system (E_1) (see Lemma 3.2). Thus, if we take $z \in L_0 \cap \mathcal{N}_c$, the solution $(x(t), y(t))$ departing from z moves along L_0 in the clockwise sense and cannot reach the origin at any finite time. Accordingly, as long as we run system (E_1) we have $\theta(t, z) > -\pi$, for every $t \geq 0$. This proves the first assertion of the lemma. On the other hand, if $z \in L_c \cap \mathcal{N}_c$, for $c < 0$ (in particular $z \notin L_0$) we know that z belongs to a closed orbit of system (E_1) whose fundamental period $\tau(z)$ tends to zero as $\mu = n_1$ grows to infinity (see Lemma 3.4). As a consequence, during a time interval of length $\alpha > 0$, the trajectory will make at least $\lfloor \frac{\alpha}{\tau(z)} \rfloor$ turns around P_1. From this fact, the second assertion of the lemma easily follows. $\square$

We have now at hand almost all the needed tools to construct a domain containing a topological horseshoe. First, however, we need to prove another technical result.

Suppose that $n_0 > 0$ is a given constant satisfying

$$n_0 < m_0^*.$$

We consider system (E_0) and look at the orbits of such a system passing through a point $(p_0, 0)$ with $0 < p_0 < 1$. Recall also the definition of the function σ in Lemma 3.2 that we call now

$$\sigma_0(p_0, \xi) := \int_{p_0}^{\xi} \frac{ds}{\sqrt{g(s^2 - p_0^2) - 2n_0(\mathcal{F}(s) - \mathcal{F}(p_0))}}, \quad \xi \in]p_0, 1].$$

The level line L_c^0 ($c := \mathcal{E}_0(p_0, 0)$) crosses the vertical line $x = \xi$ exactly at two points and $\sigma_0(p_0, \xi)$ is half of the time needed to move from one of these intersection points to the other along the orbit of (E_0) and passing through $(p_0, 0)$.

First of all, we observe that there is a $\check{p}_0 \in]0, a[$ such that

$$\sigma_0(p_0, a) > \frac{\beta - \alpha}{2}, \quad \text{for every } p_0 \in]0, \check{p}_0]. \tag{12}$$

Next, assuming (H_0), we consider system (E_1) for a general $n_1 > m_1^*$ and consider a level line L_c^1 ($c := \mathcal{E}_1(p_0, 0)$) passing through a point $(p_0, 0)$ with

$$0 < p_0 < a.$$

We denote by $(p_1, 0)$ with

$$a_{n_1} < p_1 < b_{n_1}$$

the other intersection of such a level line with the x-axis. Note that p_1 depends by n_1 even if, for sake of simplicity in the notation, we do not make this fact explicit.

We claim that there exist $\hat{p}_0 \in]0, a[$ and $m_2^* > m_1^*$ such that if we choose

$$p_0 \in]0, \hat{p}_0[$$

and

$$n_1 > m_2^*,$$

then

$$\sigma_0(p_1, b_{n_1}) < \frac{\beta - \alpha}{2}. \tag{13}$$

To prove this claim, we need to introduce some further notation. By the sign conditions on $F(s)$ and (H_0), we know that there exists a (unique) value

$$b \in]a, 1[$$

such that

$$\int_0^b F(s) \, ds = 0.$$

With this position we have that

$$b_\mu > b, \quad \forall \mu > m_1^*$$

and $b_\mu \to b$ as $\mu \to +\infty$ with the function $\mu \to b_\mu$ decreasing on $]m_1^*, +\infty)$. Thus, if we fix a value

$$\bar{\mu} > m_1^*,$$

we can conclude that

$$b \le b_\mu \le b_{\bar{\mu}}, \quad \forall \mu \ge \bar{\mu}.$$

Hence, we can define

$$\eta := \min_{s \in [b, b_{\bar{\mu}}]} F(s) > 0.$$

Let us take $n_1 = \mu \ge \bar{\mu}$ and a point $p_0 \in]0, a[$ which defines a corresponding point p_1 as described above. The proof of our claim starts now.

As a consequence of Lemma 3.3 it will be sufficient to prove that

$$\frac{b_{n_1}}{p_1} < \cosh\left(\frac{\beta - \alpha}{2}\sqrt{g - n_0 \Lambda}\right) := \kappa,$$

that is, we want to prove the inequality

$$p_1 > \frac{b_{n_1}}{\kappa}. \tag{14}$$

If $\frac{b_{n_1}}{\kappa} \le a_{n_1}$ we are done. Hence, it is not restrictive if, from now on, we suppose the opposite inequality. As a consequence we have that (14) is satisfied if and only if the level line passing through $(\frac{b_{n_1}}{\kappa}, 0)$ has energy less than $\mathcal{E}_1(p_0, 0)$.

To this aim, we first prove that there is a lower bound δ_0 (depending on n_1) such that

$$\frac{g}{2}\left(\frac{b_{n_1}}{\kappa}\right)^2 - n_1 \mathcal{F}\left(\frac{b_{n_1}}{\kappa}\right) \ge \delta_0 > 0. \tag{15}$$

Using the fact that $\mathcal{E}_1(b_{n_1}, 0) = 0$ (by definition of b_{n_1}), the last assertion is equivalent to

$$n_1\left(\mathcal{F}(b_{n_1}) - \mathcal{F}\left(\frac{b_{n_1}}{\kappa}\right)\right) - \frac{g}{2}b_{n_1}^2\left(1 - \frac{1}{\kappa^2}\right) \ge \delta_0.$$

We distinguish two cases, according to the fact that $\frac{b_{n_1}}{\kappa} > b$ or $\frac{b_{n_1}}{\kappa} \le b$.

In the former case, we can write

$$n_1\left(\mathcal{F}(b_{n_1}) - \mathcal{F}\left(\frac{b_{n_1}}{\kappa}\right)\right) - \frac{g}{2}b_{n_1}^2\left(1 - \frac{1}{\kappa^2}\right) \ge n_1 \eta\, b_{n_1}\frac{\kappa - 1}{\kappa} - \frac{g}{2}b_{n_1}^2\left(1 - \frac{1}{\kappa^2}\right)$$

$$\ge n_1 \eta\, b\frac{\kappa - 1}{\kappa} - \frac{g}{2}\left(1 - \frac{1}{\kappa^2}\right) := \delta_0,$$

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with $\delta_0 > 0$ provided that

$$n_1 > \mu^* := \frac{g(\kappa + 1)}{2\eta b} \, .$$

In the latter case, we have

$$n_1 \left(\mathcal{F}\big(b_{n_1}\big) - \mathcal{F}\Big(\frac{b_{n_1}}{\kappa}\Big) \right) - \frac{g}{2} b_{n_1}^2 \, (1 - \tfrac{1}{\kappa^2}) \geq n_1 \, \mathcal{F}\big(b_{n_1}\big) - \frac{g}{2} b_{n_1}^2 \, (1 - \tfrac{1}{\kappa^2})$$

$$= n_1 \left(\mathcal{F}\big(b_{n_1}\big) - \mathcal{F}(b) \right) - \frac{g}{2} b_{n_1}^2 \, (1 - \tfrac{1}{\kappa^2})$$

$$\geq n_1 \, \eta \, b \frac{\kappa - 1}{\kappa} - \frac{g}{2} \, (1 - \tfrac{1}{\kappa^2}) = \delta_0 \, .$$

Thus, it remains to find p_0 such that $\mathcal{E}_1(p_0, 0) \in \,]-\delta_0, 0[\,$. To check this fact, it is equivalent to prove that

$$-\mathcal{E}_1(p_0, 0) = -n_1 \mathcal{F}(p_0) + \frac{g}{2} \, p_0^2 < n_1 \, \eta \, b \frac{\kappa - 1}{\kappa} - \frac{g}{2} \, (1 - \tfrac{1}{\kappa^2}),$$

that is,

$$n_1 \left(\eta \, b \frac{\kappa - 1}{\kappa} + \mathcal{F}(p_0) \right) > \frac{g}{2} \left(p_0^2 + \tfrac{\kappa^2 - 1}{\kappa^2} \right). \tag{16}$$

We define $\hat{p}_0 \in \,]0, a[$ as the solution of equation

$$\mathcal{F}(s) = -\eta \, b \frac{\kappa - 1}{2\kappa}$$

and take p_0 such that

$$0 < p_0 \leq \hat{p}_0 \, .$$

For n_1 satisfying

$$n_1 > \tilde{\mu} := \frac{g(\kappa^2(a^2 + 1) - 1)}{\kappa(\kappa - 1)b\eta}$$

we obtain

$$n_1 \left(\eta \, b \frac{\kappa - 1}{\kappa} + \mathcal{F}(p_0) \right) \geq n_1 \, \eta \, b \frac{\kappa - 1}{2\kappa} > \frac{g}{2} \left(a^2 + \tfrac{\kappa^2 - 1}{\kappa^2} \right)$$

and (16) is proved. In this manner, we have found $\hat{p}_0$ and our claim is proved for

$$n_1 > m_2^* := \max\{\mu^*, \tilde{\mu}\}.$$

We are now in position to construct a suitable domain including a topological horseshoe. This is discussed in the next section.

4 Proof of the main result

Assume (H_0) and let $n(t)$ be like in (11), with

$$0 < n_0 < m_0^* < m_2^* < n_1 .$$

Let us fix also a number

$$\bar{p}_0 \in \,]0, p^*[, \quad \text{with } \; p^* := \min\{\check{p}_0, \hat{p}_0\}$$

and consider the compact annular region

$$\mathcal{M}_c^1 := \{(x, y) : 0 \le x \le 1, \; c \le \mathcal{E}_1(x, y) \le 0\}, \quad \text{for } c = \mathcal{E}_1(\bar{p}_0, 0).$$

As in Section 3 (see, in particular, Lemma 3.5) we consider also the set

$$\mathcal{N}_c := \mathcal{M}_c^1 \cap \big([a_{n_1}, +\infty) \times [0, +\infty) \big).$$

Now we introduce a polar coordinate system with center at $P_1 = (a_{n_1}, 0)$ and consider the following compact subsets $\mathcal{N}_c'$ and $\mathcal{N}_c''$ defined by

$$\mathcal{N}_c' := \{z \in \mathcal{N}_c \, : \, \theta(\alpha, z) \in [-\frac{5\pi}{2}, -2\pi] \}$$

and

$$\mathcal{N}_c'' := \{z \in \mathcal{N}_c \, : \, \theta(\alpha, z) \in [-\frac{9\pi}{2}, -4\pi] \},$$

respectively.

Notice that a point z belongs to $\mathcal{N}_c'$ if and only if the solution $(x(t), y(t))$ of (E_1) with $(x(0), y(0)) = z \in \mathcal{N}_c$ is such that

$$(x(\alpha), y(\alpha)) \in \mathcal{M}_c^1 \cap \big([a_{n_1}, +\infty) \times (-\infty, 0] \big) \tag{17}$$

and $x(\cdot)$ has exactly two strict maximum points separated by one strict minimum point along the time interval $[0, \alpha]$, that is the trajectory crosses the $x-$axis exactly three times. On the other hand, z belongs to $\mathcal{N}_c''$ if and only if the solution $(x(t), y(t))$ of (E_1) with $(x(0), y(0)) = z \in \mathcal{N}_c$ satisfies (17) and $x(\cdot)$ has exactly three strict maximum points separated by two strict minimum points along the time interval $[0, \alpha]$, that is the trajectory crosses the $x-$axis exactly five times.

Remark 4.1 *For sake of simplicity in the exposition we have confined ourselves to the construction of only two sets $\mathcal{N}_c'$ and $\mathcal{N}_c''$ in order to obtain at the end a dynamics on two symbols. We point out, however, that Lemma 3.5 permits to define in the same manner (if one takes n_1 large enough) an arbitrary number ℓ of pairwise disjoint sets and, correspondingly, to obtain at the end a semiconjugation with a Bernoulli shift on ℓ symbols.*

Let $\gamma : [0, 1] \to \mathbb{R}^2$ be a continuous mapping such that $\bar{\gamma}$ crosses $\mathcal{N}_c$ from its inner boundary L_c^1 to its outer boundary L_0^1. Without loss of generality (if necessary, we can restrict γ to a closed subinterval of its domain), we assume that

$$\mathcal{E}_1(\gamma(0)) = c < 0 = \mathcal{E}_1(\gamma(1)), \quad \gamma(t) \in \mathcal{N}_c, \ \forall\, t \in [0, 1]. \tag{18}$$
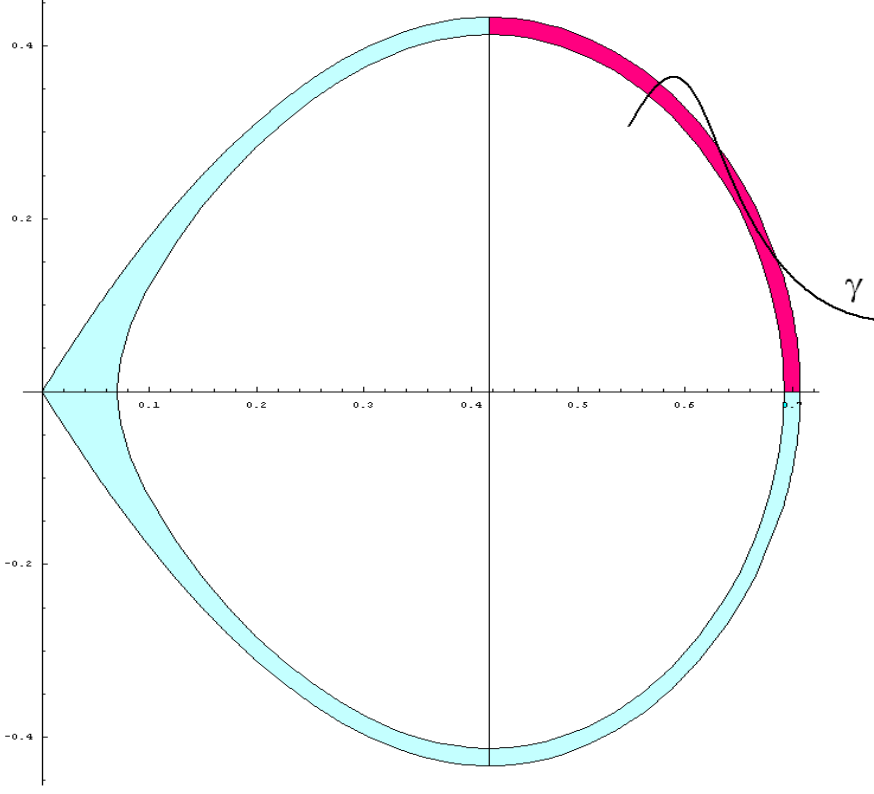

Fig. 3. *Example of a region $\mathcal{M}_c^1$ in which we have put in evidence the subset $\mathcal{N}_c$. The chosen parameters are the same like in Figure 2. We have taken as the inner boundary of $\mathcal{M}_c^1$ the level line passing through the point $(0.07, 0)$ yielding to $c \simeq -0.00850436$. In this case we have $a_{n_1} \simeq 0.417157 > a = 0.4$ and $\mathcal{E}_1(a_{n_1}, 0) \simeq -0.0936779$. A path γ crossing the inner and the outer boundaries of the set $\mathcal{N}_c$ is also shown. For our argument we consider only the restriction of γ to a subinterval of its domain in order to have condition (18) satisfied.*

Lemma 3.5 guarantees the existence of $n^* = n_{2,c}^*$ such that for $n_1 > n^*$,

$$\theta(\alpha, z) < -\frac{9\pi}{2}, \quad \forall\, z \in L_c \cap \mathcal{N}_c,$$

follows. On the other hand,

$$\theta(\alpha, z) > -\pi, \quad \forall\, z \in L_0 \cap \mathcal{N}_c.$$

If we consider now the composite map

$$[0, 1] \ni t \mapsto \theta(\alpha, \gamma(t)) \in \mathbb{R},$$

we obtain

$$\theta(\alpha, \gamma(0)) < -\frac{9\pi}{2} < -\pi < \theta(\alpha, \gamma(1)).$$

A standard continuity argument allows to determine two subintervals $[t_1'', t_2'']$ and $[t_1', t_2']$ of $[0, 1]$ with $0 < t_1'' < t_2'' < t_1' < t_2' < 1$, such that

$$\gamma(t) \in \mathcal{N}_c', \ \forall\, t \in [t_1', t_2'], \qquad \gamma(t) \in \mathcal{N}_c'', \ \forall\, t \in [t_1'', t_2'']$$

and, moreover,

$$\theta(\alpha, \gamma(t_1')) = -\frac{5\pi}{2}, \ \theta(\alpha, \gamma(t_2')) = -2\pi,$$

(19)

$$\theta(\alpha, \gamma(t_1'')) = -\frac{9\pi}{2}, \ \theta(\alpha, \gamma(t_2'')) = -4\pi.$$

Let now p_0 be an arbitrary, but fixed number such that

$$\bar{p}_0 < p_0 < p^*.$$

We denote by $\mathcal{W}$ the part of the plane between the level lines for energy $\mathcal{E}_0$ of system (E_0) passing, respectively, through $(p_0, 0)$ and $(p_1, 0)$, where we recall that $(p_1, 0)$ is other intersection of the level line

$$\mathcal{E}_1(x, y) = \mathcal{E}_1(p_0, 0), \quad 0 \le x \le 1,$$

with the $x-$axis (see Figure 4 below).

The intersection of $\mathcal{M}_c^1$ with $\mathcal{W}$ defines two connected components

$$\mathcal{M}_c^1 \cap \mathcal{W} = \mathcal{A} \cup \mathcal{B},$$

where we have denoted by $\mathcal{A}$ the component contained in $\mathcal{N}_c$ and by $\mathcal{B}$ its symmetric part with respect to the $x-$axis. The sets $\mathcal{A}$ and $\mathcal{B}$ are generalized rectangles, that is they are homeomorphic to the unit square $[0, 1]^2$.
In principle, the set $\mathcal{A}$ (which is the upper component of $\mathcal{M}_c^1 \cap \mathcal{W}$) is not necessarily contained in $\mathcal{N}_c$. However, such inclusion occurs when n_1 is sufficiently large, as we have required throughout all the paper. To check this claim, let us consider the intersection of an arbitrary level line

$$\mathcal{E}_0(x, y) = \mathcal{E}_0(s, 0), \quad \text{for } s \in [p_0, p_1]$$
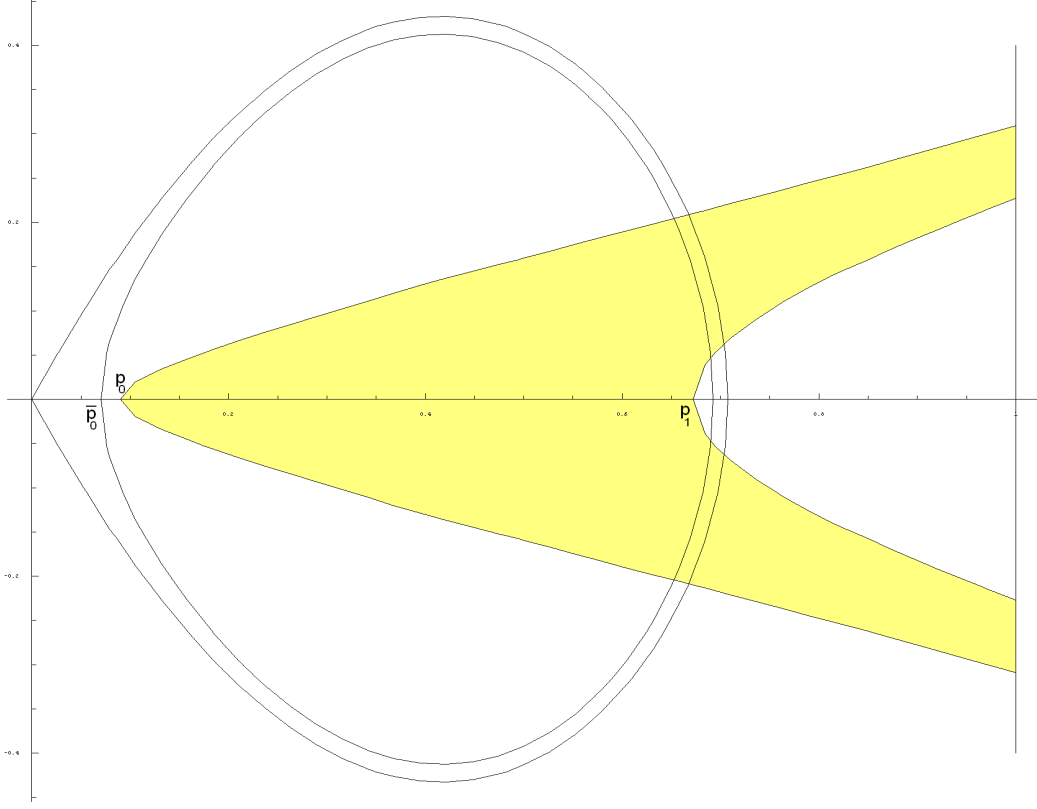
Fig. 4. *Example of a region $\mathcal{W} \subseteq [0,1] \times \mathbb{R}$ between the two level lines of energy $\mathcal{E}_0$ passing, respectively, through $(p_0,0)$ and $(p_1,0)$.*

with a level line

$$\mathcal{E}_1(x,y) = \mathcal{E}_1(\xi,0), \quad \text{for } \xi \in [0,\bar{p}_0]$$

within the strip $[0,1] \times \mathbb{R}$. By a symmetry argument, it is sufficient to look at the intersection points (x,y) with $y > 0$. This, in turns, yields to the comparison between the two functions

$$\zeta_0(x,s) := -\frac{1}{2}gs^2 + n_0\mathcal{F}(s) + \frac{1}{2}gx^2 - n_0\mathcal{F}(x)$$

and

$$\zeta_1(x,\xi) := -\frac{1}{2}g\xi^2 + n_1\mathcal{F}(\xi) + \frac{1}{2}gx^2 - n_1\mathcal{F}(x).$$

More precisely, our aim is to prove that

$$\zeta_1(x,\xi) - \zeta_0(x,s) > 0, \quad \forall\, x \in [p_0, a_{n_1}],\ \xi \in [0,\bar{p}_0],\ s \in [p_0, p_1]. \tag{20}$$

Indeed, to check (20) it is sufficient to verify that

$$\zeta(x) := \zeta_1(x,\bar{p}_0) - \zeta_0(x,p_0) > 0, \quad \forall\, x \in [p_0, a_{n_1}]. \tag{21}$$

By the above positions and taking into account that $\mathcal{F}$ is strictly decreasing on $[0, a]$, we have

$$\zeta(x) = \frac{1}{2}g({p_0}^2 - \bar{p}_0^2) + n_0(\mathcal{F}(\bar{p}_0) - \mathcal{F}(p_0)) + (n_0 - n_1)\Big(\mathcal{F}(x) - \mathcal{F}(\bar{p}_0)\Big)$$
$$\geq \frac{1}{2}g({p_0}^2 - \bar{p}_0^2) + (n_0 - n_1)\int_{\bar{p}_0}^{x} F(u)\, du$$

and therefore, we can conclude that

$$\zeta(x) \geq \frac{1}{2}g({p_0}^2 - \bar{p}_0^2) > 0, \quad \forall\, x \in [p_0, \bar{p}_0^{+}],$$

where $\bar{p}_0^{+}$ is defined as the unique point satisfying

$$\int_{\bar{p}_0}^{\bar{p}_0^{+}} F(u)\, du = 0, \quad \bar{p}_0^{+} \in\,]a, 1[\,.$$

Hence (21) is satisfied provided that we have

$$a_{n_1} \leq \bar{p}_0^{+}\,.$$

This latter property is true for n_1 sufficiently large. In fact, once that $\bar{p}_0$ is fixed, we have that $\bar{p}_0^{+} > a$ is fixed too and we already observed that $a_\mu \to a$ as $\mu \to +\infty$. Thus, having proved our claim, we know that the generalized rectangles $\mathcal{A}$ and $\mathcal{B}$ look like in Fig. 5.

We fix now an orientation for sets $\mathcal{A}$ and $\mathcal{B}$. More precisely, given a generalized rectangle $\mathcal{R}$ (also called a two-dimensional cell) we select on its boundary two disjoint compact arcs $\mathcal{R}_l^-$ and $\mathcal{R}_r^-$, like in Section 2. Such arcs will be conventionally called the *left* and the *right* sides of $\mathcal{R}$. We also set $\mathcal{R}^- := \mathcal{R}_l^- \cup \mathcal{R}_r^-$.
In the case of the sets $\mathcal{A}$ and $\mathcal{B}$, we put

$$\mathcal{A}^- := \mathcal{A} \cap \partial \mathcal{M}_c^1, \quad \mathcal{B}^- := \mathcal{B} \cap \partial \mathcal{W}.$$

Accordingly, we denote by $\mathcal{A}_l^-$ and $\mathcal{A}_r^-$ the two components of $\mathcal{A}^-$ and, similarly, by $\mathcal{B}_l^-$ and $\mathcal{B}_r^-$ the two components of $\mathcal{B}^-$ (the order is not relevant). At last, we also define the disjoint compact sets

$$\mathcal{D}_1 := \mathcal{A} \cap \mathcal{N}_c', \quad \mathcal{D}_2 := \mathcal{A} \cap \mathcal{N}_c''. \tag{22}$$

Let $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ be the Poincaré operator associated to system (7), that is $z \mapsto \zeta(\beta, z)$, where $\zeta(t) = \zeta(t, z)$ is the solution of system (7) with $\zeta(0) = z$. Observe that
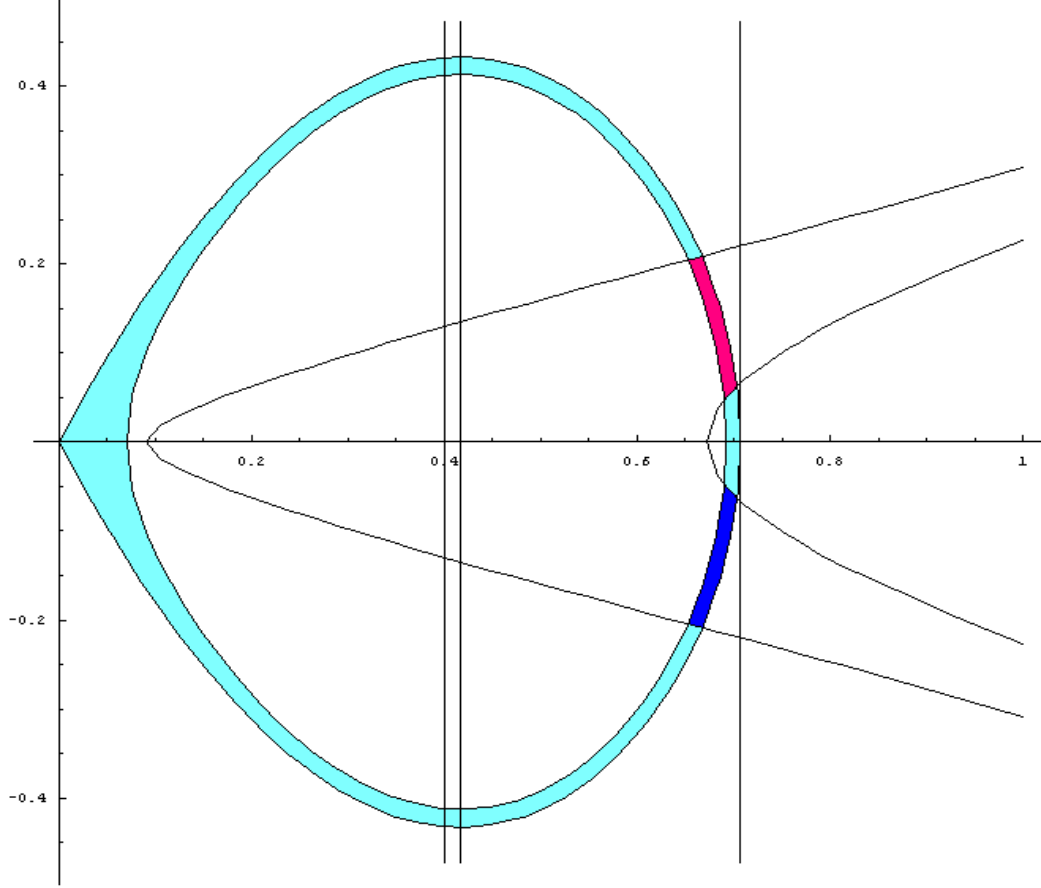
$$\psi = \psi_0 \circ \psi_1,$$

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Fig. 5. *Example of a region $\mathcal{M}_c^1$ in which we have put in evidence the intersection with $\mathcal{W}$ consisting of two generalized rectangles $\mathcal{A}$ (the upper component) and $\mathcal{B}$ (the lower component). The parameters are the same like in Figure 2 and Figure 3. Moreover, we have also chosen $\bar{p}_0 = 0.07$ and obtained $\bar{p}_0{}^+ \simeq 0.652494$. In the picture we have drawn the vertical lines $x = a = 0.4$ and $x = a_{n_1} \simeq 0.417157$ to show that $\mathcal{A} \subseteq \mathcal{N}_c$.*

where ψ_1 is the Poincaré map associated to system (E_1) for the time interval $[0, \alpha]$ and ψ_0 is the Poincaré map associated to system (E_0) for the time interval $[0, \beta - \alpha]$.

As a consequence of (19) and the fact that the annulus $\mathcal{M}_c^1$ is invariant with respect to the dynamical system associated to (E_1), we have

$$(\mathcal{D}_i, \psi_1) : (\mathcal{A}, \mathcal{A}^-) \looparrowleft (\mathcal{B}, \mathcal{B}^-), \quad \text{for } i = 1, 2.$$

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On the other hand, we also have that

$$(\mathcal{B}, \psi_0) : (\mathcal{B}, \mathcal{B}^-) \looparrowleft (\mathcal{A}, \mathcal{A}^-).$$

Thus, we can conclude that

$$(\mathcal{D}_i, \psi) : (\mathcal{A}, \mathcal{A}^-) \looparrowleft (\mathcal{A}, \mathcal{A}^-), \quad \text{for } i = 1, 2.$$

The results in [11] and [15], [16] guarantee the existence of a compact invariant set

$$\Lambda \subseteq \mathcal{D}_1 \cup \mathcal{D}_2 \subseteq \mathcal{A}$$

for which $\psi|_\Lambda$ is semiconjugate to a two-sided Bernoulli shift. Moreover, for each periodic sequence of two symbols there is a point in Λ which is periodic. In fact, all the assumptions of Lemma 2.1 are satisfied with respect to the Poincaré map ψ, the oriented rectangle $(\mathcal{R}, \mathcal{R}^-) := (\mathcal{A}, \mathcal{A}^-)$ and the sets $\mathcal{D}_i$ $(i = 1, 2)$ defined in (22).

As a final step, we want to give an interpretation of the above statement in terms of the solutions to equation

$$x'' - gx + n(t)F(x) = 0.$$

Consider a two-sided sequence of two symbols $(i_k)_{k \in \mathbb{Z}}$ with

$$i_k \in \{1, 2\}, \quad \forall\, k \in \mathbb{Z}.$$

By the semiconjugation, we know that there is at least one point $(x_0, y_0) = z_0 \in \Lambda$ such that

$$\psi^k(z_0) \in \mathcal{D}_{i_k}, \quad \forall\, k \in \mathbb{Z}.$$

Let $x(\cdot)$ (globally defined on $\mathbb{R}$) be the solution of the differential equation with $x(0) = x_0$ and $x'(0) = y_0$. For such a solution, we have that

$$(x(k\beta), x'(k\beta)) \in \mathcal{D}_{i_k}, \quad \forall\, k \in \mathbb{Z}. \tag{23}$$

By construction,

$$0 < x < 1, \quad \forall\, (x, y) \in \mathcal{D}_1 \cup \mathcal{D}_2.$$

Hence, Lemma 3.1 ensures that

$$0 < x(t) < 1, \quad \forall\, t \in \mathbb{R}$$

and thus we have that $x(\cdot)$ is a solution of the original equation (6) no matter which kind of modification has been performed on F outside the interval $[0, 1]$.

Now, we explain the meaning of (23) by analyzing the following two possibilities:

- $(x(k\beta), x'(k\beta)) \in \mathcal{D}_1$, for some $k \in \mathbb{Z}$;
- $(x(k\beta), x'(k\beta)) \in \mathcal{D}_2$, for some $k \in \mathbb{Z}$.

In the former case, $x(t)$ has precisely two strict maximum points separated by one strict minimum point along the time interval $[(k-1)\beta, (k-1)\beta+\alpha]$, while, in the latter case, $x(t)$ has precisely three strict maximum points separated by two strict minimum points along the time interval $[(k-1)\beta, (k-1)\beta+\alpha]$. In both the situations, $x(\cdot)$ is convex in the interval $[(k-1)\beta + \alpha, k\beta]$, with $x'((k-1)\beta + \alpha) < 0$ and $x'(k\beta) > 0$.

At last, we observe that if the sequence $(i_k)_{k \in \mathbb{Z}}$ is periodic, that is $i_k = i_{k+\ell}$ for some $\ell \geq 1$, then we can take $x(\cdot)$ as a $\ell\beta$-periodic solution as well.

The proof of our main theorem is complete. $\qquad\square$

The proof of Theorem 1.2 relies on slight modifications of the arguments described above and therefore it is omitted.

Both Theorem 1.1 and Theorem 1.2 are stable with respect to small (in the L^1-norm) perturbations of the weight function $n(\cdot)$.

5 Appendix

In this section we give a proof of Lemma 3.1. Actually, we present a more general result which improves [20, Lemma 2.1] and may have some independent interest.

We consider the second order scalar ODE

$$x'' + h(t, x) = 0 \tag{24}$$

where $h : [t_0, t_1] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, we assume that $h(\cdot, s)$ is measurable for all $s \in \mathbb{R}$, $h(t, \cdot)$ is continuous for almost every $t \in [t_0, t_1]$ and, for every $r > 0$ there is a measurable function $\rho_r \in L^1([t_0, t_1], \mathbb{R}^+)$ such that $|h(t, s)| \leq \rho_r(t)$ for almost every $t \in [t_0, t_1]$ and every $s \in [-r, r]$. Solutions of (24) are considered in the Carathéodory sense too (cf. [7, p.28]).

Lemma 5.1 *Suppose*

$$h(t, 0) \equiv 0$$

and, for a.e. $t \in [t_0, t_1]$,

$$h(t, s) > 0, \ \forall\, s < 0, \qquad h(t, s) < 0, \ \forall\, s \geq 1.$$

Let $x(\cdot)$ be a solution of (24) defined on $[t_0, t_1]$ and such that

$$0 < x(t_0), x(t_1) < 1.$$

Then $0 \leq x(t) \leq 1, \forall t \in [t_0, t_1]$. Moreover, $x(t) > 0, \forall t \in [t_0, t_1]$ if h is (locally) lipschitzian at $s = 0$ and $x(t) < 1, \forall t \in [t_0, t_1]$, if h is (locally) lipschitzian at $s = 1$.

Proof. At first we prove that $x(t) \geq 0$ for all $t \in [t_0, t_1]$. If, by contradiction $x(\tilde{t}) < 0$ for some $\tilde{t} \in [t_0, t_1]$, then we can find σ_0, σ_1 with $t_0 < \sigma_0 < \tilde{t} < \sigma_1 < t_1$ such that $x(\sigma_0) = x(\sigma_1) = 0$ and $x(t) < 0$ for all $t \in]\sigma_0, \sigma_1[$. Multiplying equation

$$-x''(t) = h(t, x(t)) \tag{25}$$

by $x(t)$ and then integrating on $[\sigma_0, \sigma_1]$, we obtain $\int_{\sigma_0}^{\sigma_1} x'(t)^2 \, dt = \int_{\sigma_0}^{\sigma_1} h(t, x(t)) x(t) \, dt < 0$, a contradiction. A similar computation shows that $v(x) \leq 1$ for all $x \in [t_0, t_1]$. In fact, if by contradiction $x(\hat{t}) > 1$ for some $\hat{t} \in [t_0, t_1]$, then we can find τ_0, τ_1 with $t_0 < \tau_0 < \hat{t} < \tau_1 < t_1$ such that $x(\tau_0) = x(\tau_1) = 1$ and $x(t) > 1$ for all $t \in]\tau_0, \tau_1[$. Multiplying equation (25) by $x(t) - 1$ and then integrating on $[\tau_0, \tau_1]$, we obtain $\int_{\tau_0}^{\tau_1} x'(t)^2 \, dt = \int_{\tau_0}^{\tau_1} h(t, x(t))(x(t) - 1) \, dt < 0$, a contradiction.

If we have $\min x(t) = x(t_*) = 0$, then also $x'(t_*) = 0$ since $t_* \in]t_0, t_1[$. Hence $x(\cdot)$ is a solution of the initial value problem

$$x'' + h(t, x) = 0, \quad x(t_*) = x'(t_*) = 0$$

and therefore, if h satisfies a Lipschitz condition at $s = 0$, we must have $x(t) = 0, \forall t \in [t_0, t_1]$ (a contradiction with our hypotheses).

Assume now that h satisfies a Lipschitz condition at $s = 1$ and $\max x(t) = x(t^*) = 1$. In this case, $x'(t^*) = 0$ because $t^* \in]t_0, t_1[$. Define also the auxiliary function $\tilde{h}(t, s) := h(t, \max\{1, s\})$ which is lipschitzian at $s = 1$ too and let $y(t)$ be a solution of the Cauchy problem

$$x'' + \tilde{h}(t, x) = 0, \quad x(t^*) = 1, \ x'(t^*) = 0$$

defined on a maximal interval of existence. Using the fact that $\tilde{h}(t, s) < 0$ for all $s \in \mathbb{R}$ and almost every $t \in [t_0, t_1]$, we find that $y(t) > 1$ for all $t \neq t^*$ in the domain of $y(\cdot)$ and therefore $y(\cdot)$ is a solution of (24) as well. By the local uniqueness of the solutions to the Cauchy problem under consideration we conclude that $x(\cdot) = y(\cdot)$ in a neighborhood of t^* (a contradiction to $x(t) \leq 1 < y(t)$ for $t \neq t^*$). □

We remark that the same proof works for the slightly more general equation

$$x'' + cx' + h(t, x) = 0,$$

under the same assumptions on h and for every $c \in \mathbb{R}$.

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